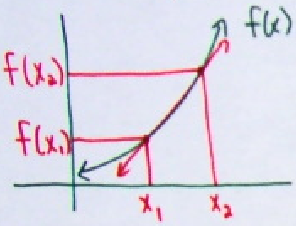
Chapter 6 Applications of the Derivative

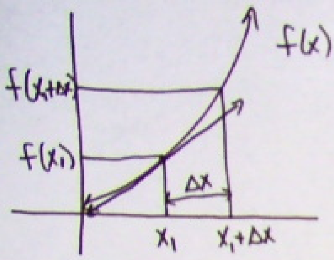
6.1 Rates of Change

Suppose y is a function of x ie: $y=f(x)$



$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ is called the <u>average</u> rate of change of y with respect to x over the interval $[x_1, x_2]$

* note: the average rate of change is the slope of the <u>secant line</u> joining x_1 and x_2.



instantaneous rate of change at x_1

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

<u>Summary</u> 1) av. rate of change of $f(x)$ between x_1 and x_2 is the slope of the secant line joining x_1 and x_2

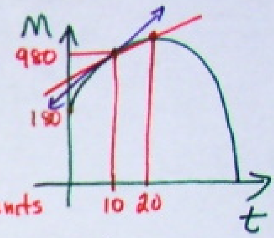
$$\text{average rate of change} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

2) the rate of change (inst) of $f(x)$ at x_1 is the slope of the tangent line at x_1

$$\text{rate of change} = f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

ex11 an employee's monthly production, in # of units produced, M, is found to be a function of the # of years of service, t.

$$M(t) = -2t^2 + 100t + 180$$



a) find the productivity at 10 years

$M(10) = -2(10)^2 + 100(10) + 180 = 980$ units

b) find the average rate of change of productivity between 10 and 20 years

$$\frac{\Delta M}{\Delta t} = \frac{M(20) - M(10)}{20 - 10} = \frac{1380 - 980}{10} = 40 \text{ \# of units/year}$$

c) find the rate of change of productivity at 10 years

$M'(t) = -4t + 100$

$M'(10) = -40 + 100 = 60$ units/year.

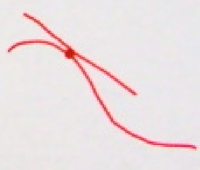
ex11 the temp, T, of a person during an illness is given by $T(t) = \frac{4t}{t^2+1} + 98.6$, where T is the temp in °F and t is the time in hours

a) find the temp at 2hrs $\quad T(2) = 100.2°F$

b) find the rate of change of the temp at 2hrs

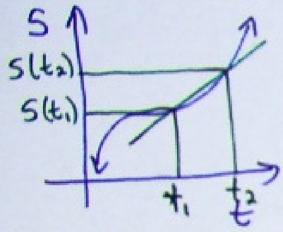
$$T'(t) = \frac{4(t^2+1) - 2t(4t)}{(t^2+1)^2} = \frac{4t^2+4-8t^2}{(t^2+1)^2}$$

$$= \frac{-4t^2+4}{(t^2+1)^2}$$



$$T'(2) = \frac{-12}{25} \ °F/hr$$

6.2 Distance, Velocity, Acceleration

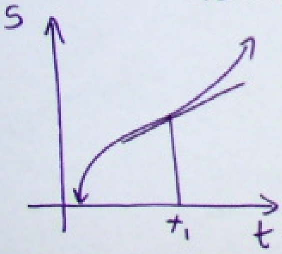


suppose the position, s, of an object was a function of time, t

ie: $s(t)$

$$\text{av rate of change} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Velocity (av) =

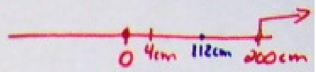


Velocity (inst) = $s'(t)$

$V(t) = s'(t)$

ex11 a ball rolls along the x-axis so that its position in cm after t seconds is

$s(t) = t^3 + 3t^2, \; t \geq 0$



a) find the V_{av} between 1 and 5 seconds

$$V_{av} = \frac{\Delta s}{\Delta t} = \frac{s(5) - s(1)}{5-1} = \frac{(125+75)-(1+3)}{4} = \frac{200-4}{4} = 49\,m/s$$

b) find the velocity at 5 seconds

$v(t) = s'(t) = 3t^2 + 6t$

$v(5) = 3(5)^2 + 6(5) = 105\,cm/s$

c) where is the ball when its velocity is 72cm/s?

$v(t) = 3t^2 + 6t$

$72 = 3t^2 + 6t$

$0 = 3t^2 + 6t - 72$

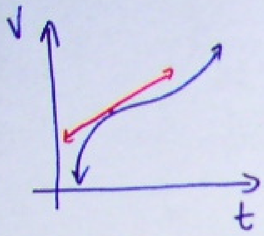
$0 = 3(t^2 + 2t - 24)$

$= 3(t-4)(t+6)$

$t = 4 \quad \cancel{t = -6}$

$s(4) = 64 + 3(16)$

$= 112cm$



$v(t)$

$v'(t) = a(t)$

$\frac{dv}{dt}$ = acceleration

* Summary given the position function $s(t)$

$v(t) = s'(t)$

$a(t) = v'(t) = s''(t)$

$s(t) = t^3 + 3t^2 , \; t \geq 0$

$v(t) = 3t^2 + 6t$

a) what is the accel of the ball when $t = 1$?

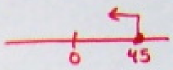
$a(t) = v'(t) = 6t + 6 \qquad a(1) = 12 \, m/s/s = m/s^2$

e) what is the vel of the ball when accel = $54 \, cm/s^2$

$54 = 6t + 6 \quad t = 8 \qquad v(8) = 240 \, m/s$

ex11 Suppose a bee flies along the x-axis so that its position (cm) at time t (sec) is given by

$$s(t) = 2t^3 - 21t^2 + 60t , \; t \geq 0$$



a) where is the bee at 3sec? $s(3) = 45cm$

b) what is the bee's velocity at 3 sec?

$v(t) = s'(t) = 6t^2 - 42t + 60 \qquad v(t) = v(3) = -12 cm/s$

c) when was the bee not moving?

$v(t) = 0 \quad 6t^2 - 42t + 60 = 0$

$6(t^2 - 7t + 10) = 0$

$6(t-5)(t-2) = 0$

$t = 5sec$

$t = 2sec$

d) when was the bee moving to the right?

$v(t) > 0$

$6(t-5)(t-2) > 0$

$[0, 2) \cup (5, \infty)$

$[- - - - - - 0 + + + \; (t-5)$

$[- - 0 + + + + + + + \; (t-2)$

$0 \quad 2 \quad 5$

$[+ + + \; 0 - - - \; 0 + + + \;)(t-5)(t-2)$

ex// a rock is thrown upwards from the top of a house and its height above ground (m) after t seconds is

$$h(t) = -5t^2 + 40t + 5$$

a) find the initial height

$h(0) = 5m$

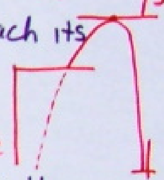
b) find velocity at 2sec

$V(t) = h'(t) = -10t + 40 \quad V(2) = 20 m/s$

c) find accel at 2sec

$a(t) = v'(t) = -10 m/s^2$

d) when does the rock reach its max height?



$v(t) = 0 = -10t + 40$

$t = 4sec$

e) what is the max height?

$h(4) = 85m$

f) with what velocity does the rock hit the ground?

$h = 0 = -5t^2 + 40t + 5$

$0 = t^2 - 8t - 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{8 \pm \sqrt{64 - 4(1)(-1)}}{2} = \frac{8 \pm \sqrt{68}}{2} \approx 8.1 \text{ sec}, \ \cancel{-.12 \text{ sec}}$$

$V(8.1) = -10(8.1) + 40$

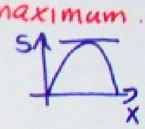
$-39 m/s$

6.3 Optimization Problems

* read pg 273 - steps for problem solving

review 5.2 pg 224 finding absolute extrema

ex// 2 non-negative numbers have a sum of 20. Find these numbers if the sum of their product and four times one of the numbers is a maximum. what is the max sum?



understand the problem

$2 + 18 \qquad (2)(18) + 4(18) = 108$

$5 + 15 \qquad (5)(15) + 4(15) = 135$

$10 + 10$

$18 + 2$

let x = one number

$20 - x$ = other #

$S = (x)(20-x) + 4x$

$= 20x - x^2 + 4x$

$S(x) = -x^2 + 24x$, $[0, 20]$

$S'(x) = -2x + 24 = 0$

$x = 12$

12 and 8

critical #'s

$S(12) = 144$

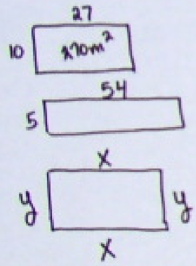
$S(0) = 0$

$S(20) = -(20)^2 + 24(20)$

$= 80$

Area Problems

ex// a farmer must enclose a rectangular pen having an area of $270\,m^2$. Material for the north and south sides cost \$10/m, while material for the east and west sides cost \$3/m. Find the dimensions that will minimize the cost of construction.



$xy = 270$

$y = \frac{270}{x}$

minimize cost $C = 10x + 10x + 3y + 3y$

$C = 20x + 6y$

$C = 20x + 6\left(\frac{270}{x}\right)$

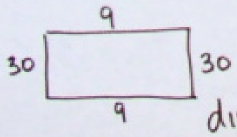
$C(x) = 20x + 1620x^{-1}$, $(0, \infty)$

$C'(x) = 20 - \frac{1620}{x^2} = 0$

$\frac{20}{1} = \frac{1620}{x^2}$ $\quad 20x^2 = 1620$

$x^2 = 81$

$x = \pm 9 = 9$

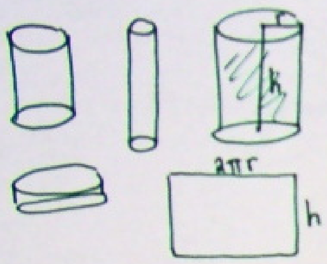


dimensions 9m x 30m $\quad y = \frac{270}{9} = 30$

$C(9) = 20(9) + \frac{1620}{9} = \$360$

Volume Problems

a pop can is to hold 400ml. Find the dimensions of the can if the amount of material is to be minimized.



$V = \pi r^2 h = 400\,cm^3$

$h = \frac{400}{\pi r^2}$

$h = \frac{400}{\pi(4)^2}$

minimize $S = 2\pi r^2 + 2\pi r h$

$S = 2\pi r^2 + 2\pi r\left(\frac{400}{\pi r^2}\right)$

$S(r) = 2\pi r^2 + 800r^{-1}$, $(0, \infty)$

$S'(r) = 2(2\pi r)$

$4\pi r - \frac{800}{r^2} = 0 \qquad 4\pi r = \frac{800}{r^2}$

$4\pi r^3 = 800$

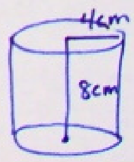
$r^3 = \frac{200}{\pi}$

$r = \sqrt[3]{\frac{200}{\pi}}$

$\approx 4cm$

$h = 7.95$

$\approx 8cm$



6.4 Related Rates - Part One

*must review implicit differentiation 4.10

recall

$$y = x^2 \qquad\qquad y = x^2$$

$$\frac{d(y)}{dx} = \frac{d(x^2)}{dx} \qquad\qquad \frac{d(y)}{dt} = \frac{dy}{dx}\cdot\boxed{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = 2x \qquad\qquad \frac{dy}{dt} = 2x\cdot\frac{dx}{dt}$$

$$y = x^2$$

find $\frac{dy}{dw} = 2x\cdot\frac{dx}{dw}$ find $\frac{dy}{dp} = 2x\frac{dx}{dp}$

ex1 differentiate $x^2 + y^2 = z^2$ with respect t

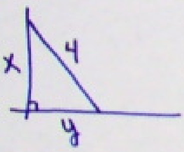
$$\frac{d(x^2)}{dt} + \frac{d(y^2)}{dt} = \frac{d(z^2)}{dt}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\cdot\frac{dz}{dt}$$

* pg282 steps

Distance and Length Problems

ex1 a 4m ladder is leaning against a building. the base of the ladder begins to slide away from the wall at a rate of $\frac{1}{2}$ m/s. At what rate is the top of the ladder moving down the wall when the base of the ladder is 2m from the wall?



$\frac{dy}{dt} = \frac{1}{2}$ find $\left.\frac{dx}{dt}\right|_{y=2}$

$x^2 + y^2 = 16$ differentiate with respect to time

$$\frac{d(x^2)}{dt} + \frac{d(y^2)}{dt} = \frac{d(16)}{dt}$$

$$\cancel{2}x\boxed{\frac{dx}{dt}} + \cancel{2}y\frac{dy}{dt} = 0$$

$x^2 + 4 = 16$

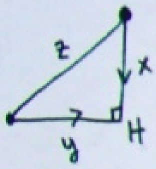
$x^2 = 12$

$x = 2\sqrt{3}$

$$(2\sqrt{3})\frac{dx}{dt} + (2)\left(\frac{1}{2}\right) = 0$$

$$\frac{dx}{dt} = \frac{-1}{2\sqrt{3}} \approx -\boxed{.289\ m/s}$$

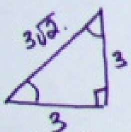
ex// 2 ships approach the same harbor, one moving south at a rate of 10km/h and the other moving east at a rate of 15km/h. How is the distance between them changing when both are 3km from the harbor?



$\frac{dx}{dt} = -10$ $\frac{dy}{dt} = -15$ find $\left.\frac{dz}{dt}\right|_{\substack{x=3\\y=3}}$

$$x^2 + y^2 = z^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\boxed{\frac{dz}{dt}}$$



$3^2 + 3^2 = z^2$

$18 = z^2$

$3\sqrt{2} = z$

$$(3)(-10) + (3)(-15) = 3\sqrt{2}\,\frac{dz}{dt}$$

$$-30 \quad -45$$

$$\frac{-75}{3\sqrt{2}} = \frac{dz}{dt} \approx -17.7 \text{ km/h}$$

6.5 Related Rates - Part Two

Area and Volume * do not need to memorize formulas

ex// the area of a circular oil slick on the surface of the sea is increasing at a rate of $150 m^2/s$. How fast is the radius changing when the radius is 25m?

A = area

r = radius

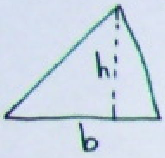
$\frac{dA}{dt} = 150$ find $\left.\frac{dr}{dt}\right|_{r=25}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \qquad 150 = 2\pi(25)\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{150}{50\pi} = \frac{3}{\pi} \approx .95 m/s$$

the base of a triangle is growing at a rate of 6cm/s while the height is shrinking at a rate of 5cm/s. what is the rate of change of the area of the triangle when the base is 20cm and the height is 16cm?



A = area $\quad \frac{db}{dt} = 6 \quad \frac{dh}{dt} = -5 \quad$ find $\left.\frac{dA}{dt}\right|_{\substack{b=20\\h=16}}$

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}\left[(1)\frac{db}{dt}h + (1)\frac{dh}{dt}b\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\left[(6)(16) + (-5)(20)\right]$$

$$\frac{1}{2}(96 - 100) = -2\,cm^2/s$$

$p \cdot q$

$p'q + q'p$

a conveyor belt at a gravel pit pours sand onto the ground at a rate of $180\,m^3/h$. the sand forms a conical pile with height one-third the diameter of the base. Find how fast the height of the pile is changing when the radius of the base is 6m.

V - volume $\quad \frac{dV}{dt} = 180 \quad$ find $\left.\frac{dh}{dt}\right|_{r=6}$

$h = \frac{1}{3}d$

$h = \frac{1}{3}(2r)$

$h = \frac{2}{3}r$

$r = \frac{3}{2}h$

$6 = \frac{3}{2}h$

$h = 4$

$$V = \frac{1}{3}\pi r^2 h \qquad V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h$$

$$V = \frac{3}{4}\pi h^3$$

$$\frac{dV}{dt} = \frac{3}{4}\pi(3h^2)\frac{dh}{dt}$$

$$180 = \frac{3}{4}\pi(3(4)^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{180}{36\pi} = \frac{5}{\pi}\,m/h$$

Chapter 6.6 - Chapter Review

1) Rates of Change (average and instantaneous)

In a certain memory experiment, a person is able to memorize M words after t minutes where:

$$M(t) = -\frac{1}{1000}t^3 + \frac{1}{10}t^2$$

a) find the average rate of change of memory between 10 and 20 minutes

$$\text{rate of change - average} = \frac{\Delta M}{\Delta t} = \frac{M(20)-M(10)}{20-10} = \frac{32-9}{10} = \frac{23}{10}$$

2.3 words/min

b) find the rate of change at 10 min.

$$M'(t) = -\frac{1}{1000}(3t^2) + \frac{1}{10}(2t)$$

$$M'(10) = -\frac{1}{\cancel{1000}}(3\cancel{00}) + \frac{1}{10}(20) = -\frac{3}{10} + 2 = 1.7 \text{ words/min}$$

2) velocity / acceleration

The motion of a particle along the x-axis is described by $s(t) = t^3 - 15t^2 + 63t$, $t \geq 0$, where t is time in seconds and s is the position in meters. Find:

a) velocity and acceleration at time t

$$v(t) = s'(t) = 3t^2 - 30t + 63 \qquad a(t) = v'(t) = 6t - 30$$

b) velocity and acceleration at 2 sec

$$v(2) = 3(2)^2 - 30(2) + 63 \qquad a(2) = 6(2) - 30$$

$$15 \text{ m/s} \qquad -18 \text{ m/s}^2$$

c) the position of the particle when its vel is 36 m/s

$$36 = 3t^2 - 30t + 63 \qquad 0 = t^2 - 10t + 9 \qquad S(1) = 49\text{m}$$

$$0 = 3t^2 - 30t + 27 \qquad (t-9)(t-1) \qquad S(9) = (9)^3 - 15(9)^2 + 63(9)$$

$$t = 9 \qquad t = 1$$

$$= 729 - 1215 + 567$$

$$= 81\text{m}$$

d) at what time(s) the particle is at rest

$$v = 0 \qquad 0 = 3t^2 - 30t + 63$$

$$0 = t^2 - 10t + 21 \qquad t = 3\text{sec}$$

$$(t-3)(t-7) \qquad t = 7\text{sec}$$

3) Optimization problems

a) number

find 2 positive numbers with a product of 200 such that the sum of one number and twice the second is as small as possible

let x = a #
y = another #

$x \cdot y = 200$
$y = \frac{200}{x}$

minimize sum S

$S = x + 2y$
$S = x + 2\left(\frac{200}{x}\right)$
$S = x + 400x^{-1}$, $(0, \infty)$
$\frac{dS}{dx} = 1 - \frac{400}{x^2} = 0$
$1 = \frac{400}{x^2}$
$x = \pm 20$

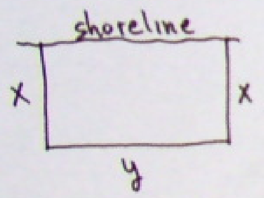
1	200
2000	$\frac{1}{10}$
$\frac{1}{1000}$	20000

$x = 20$ $\quad y = \frac{200}{20} = 10$

the #'s are 20 and 10

b) area

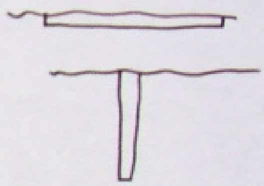
a lifeguard at a beach has 400m of rope available to layout a rectangular restricted swimming area using the straight shoreline as one side of the rectangle. what dimensions should the swimming area be to maximize its area?



$2x + y = 400$
$y = 400 - 2x$

max area A

$A = xy$
$A = x(400 - 2x)$
$A(x) = 400x - 2x^2$, $(0, 200)$
$A'(x) = 400 - 4x = 0$
$400 = 4x$
$x = 100$



4) related rate problems

a) area

the area of a triangle is increasing at a rate of $4\text{cm}^2/\text{min}$ and its base is increasing at a rate of 1cm/min. At what rate is the altitude of the triangle changing when the altitude is 20cm and the area is 80cm^2?

area

A = area
b = base
h = alt

$\frac{dA}{dt} = 4 \quad \frac{db}{dt} = 1 \quad \text{find } \left.\frac{dh}{dt}\right|_{\substack{h=20\\A=80}}$

$A = \frac{1}{2}bh$

$(1)\frac{dA}{dt} = \frac{1}{2}\left[(1)\frac{db}{dt}\cdot h + (1)\frac{dh}{dt}\cdot b\right]$

$4 = \frac{1}{2}\left[(1)(20) + \frac{dh}{dt}(8)\right]$

$8 = 20 + 8\frac{dh}{dt}$

$80 = \frac{1}{2}(b)(20)$

$b = 8$

$-12 = 8\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{12}{8} = -\frac{3}{2}$ cm/min

b) volume

Sand is being dumped on the ground at a rate of $1.2\text{m}^3/\text{min}$ and forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile growing when the pile is 3m high?

V - volume
r - radius
h - height

$\frac{dV}{dt} = 1.2 = \frac{12}{10} = \frac{6}{5} \quad \text{find } \left.\frac{dh}{dt}\right|_{h=3}$

$h = d$

$\boxed{h = 2r}$

$r = \frac{1}{2}h$

$V = \frac{1}{3}\pi r^2 h \quad V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3h^2\cdot\frac{dh}{dt}$

$\frac{6}{5} = \frac{\pi}{\cancel{12}_4}\cdot\cancel{3}(3)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{\cancel{6}^{2}(4)}{5(\cancel{9}_3\pi)}$

$= \frac{8}{15\pi}$ m/min

≈ approximate