Chapter 5

Graphical Applications of the Derivative

given a function, $f(x)$, $f'(x)$ is known as the <u>first</u> derivative of $f(x)$

5.1 Higher Order Derivatives

if $f'(x)$ can be differentiated, then $f''(x)$ is called the <u>second</u> derivative of $f(x)$

using $\frac{dy}{dx}$, the 2^{nd} derivative would be $\frac{d^2y}{dx^2}$

3^{rd}, 4^{th}, .. n^{th} derivative $\quad f'''(x) \quad \frac{d^3y}{dx^3}$

example: $f(x) = x^3 - 4x^2 + 5x - 2$

$f'(x) = 3x^2 - 8x + 5 \qquad f'''(x) = 6$

$f''(x) = 6x - 8 \qquad f''''(x) = 0$

example: $f(x) = \frac{x^2}{(3x+1)^3}$

$$f'(x) = \frac{2x(3x+1)^3 - 3(\cancel{3x+1})^2(3)(x^{\cancel{2}})}{(3x+1)^6}$$

$$= \frac{x(\cancel{3x+1})^2\left[2(3x+1) - 9x\right]}{(3x+1)^{\cancel{6}\,4}}$$

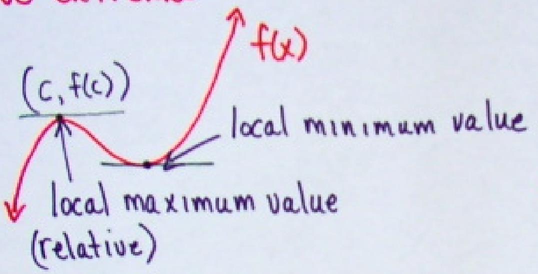
$$= \frac{x(-3x+2)}{(3x+1)^4}$$

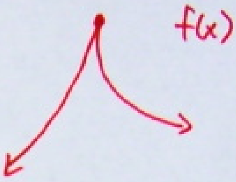
$$f'(x) = \frac{-3x^2+2x}{(3x+1)^4}$$

$$f''(x) = \frac{(-6x+2)(3x+1)^4 - 4(3x+1)^3(3)(-3x^2+2x)}{(3x+1)^8}$$

5.2 Relative and Absolute Extrema

relative extrema





Critical Number: a function, $f(x)$, has a critical number at $x=c$ if $f'(c)=0$ or $f'(c)$ does not exist and $x=c$ is in the domain of $f(x)$

example: find all critical numbers of $f(x)$

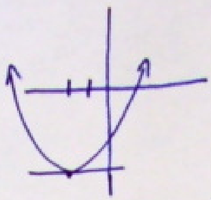
1) $f(x)=x^2+4x-11$

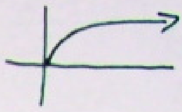
$f'(x)=2x+4=0$

$x=-2$

$f(-2)=4-8-11$

-15





2) $f(x)=\dfrac{x^2}{x-2}$

$x \neq 2$

$f'(x)=\dfrac{2x(x-2)-1(x^2)}{(x-2)^2}=0$

$f'(2)=\dfrac{-4}{0}$

$2x^2-4x-x^2=0$

$x^2-4x=0$

$x(x-4)=0$

$x=0$

$x=4$

3) $f(x)=\sqrt{x}$ $\quad x \geq 0$

$[0,\infty)$

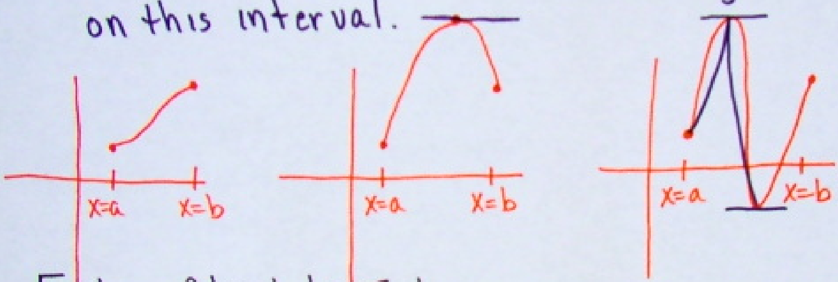
$f'(x)=\frac{1}{2}x^{-1/2}$

$=\dfrac{1}{2\sqrt{x}}$ $\quad x=0$

$\dfrac{1}{2\sqrt{x}}=0$

Absolute Extrema (global)

<u>Extreme Value theorem</u> if $f(x)$ is a continuous function on a closed interval $[a,b]$, then $f(x)$ has both a global maximum and global minimum on this interval.



Finding Absolute Extrema

1. find all critical values of $f(x)$ on $[a,b]$
2. evaluate $f(x)$ at each critical value
3. evaluate $f(x)$ at each endpoint
4. the greatest and least of these values are the global extrema

examples: Find the absolute extrema

1) $f(x) = x^2 - 4x - 1$, $[1,4]$

$f'(x) = 2x - 4 = 0$

$x = 2$

$f(1) = -4$
$f(4) = -1$
$f(2) = -5$

-1 is a global max

-5 is a global min

2) $f(x) = \dfrac{(x+1)^2}{x-1}$, $[2,4]$

$f'(x) = \dfrac{2(x+1)(x-1) - (x+1)^2}{(x-1)^2} = 0$

$(x+1)[2x-2-x-1] = 0$

$(x+1)(x-3)$

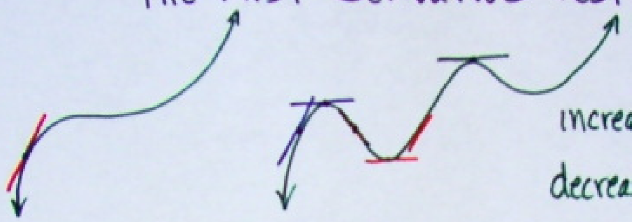
$\cancel{x=-1}$ $x = 3$

$f(2) = 9$
$f(4) = \frac{25}{3} = 8\frac{1}{3}$
$f(3) = 8$

9 is a global max

8 is a global min

5.3 Increasing and Decreasing Intervals
The First Derivative Test



increasing slope +

decreasing slope −

* if $f'(x)>0$, then $f(x)$ is increasing
$f'(x)<0$, then $f(x)$ is decreasing

<u>1st derivative test</u> if c is a critical # of a continuous function, $f(x)$, then:

- $f(x)$ has a relative minimum at $x=c$, if $f'(x)$ switches signs from neg to pos at $x=c$
- $f(x)$ has a relative maximum at $x=c$, if $f'(x)$ switches signs from pos to neg at $x=c$

example: find the open interval on which $f(x)$ is increasing or decreasing. Find the coordinates of any relative extrema

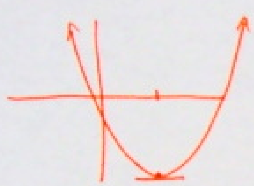
1) $f(x)=x^2-8x-20$
$f'(x)=2x-8=0$
$x=4$

1st deriv test

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	−	0	+
	\	—	/

increasing $(4,\infty)$
decreasing $(-\infty,4)$

$(4, f(4))$ local
$(4,-36)$ minimum

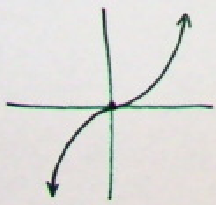


2) $f(x)=x^3$
$f'(x)=3x^2=0$
$x=0$

x	$(-\infty,0)$	0	$(0,\infty)$
$f'(x)$	+	0	+

increasing
$(-\infty,0)\cup(0,\infty)$
decreasing never

$(0,0)$



3) $f(x) = -x^4 + 18x^2 + 3$

$f'(x) = -4x^3 + 36x = 0$

$-4x(x^2 - 9)$

$-4x(x-3)(x+3) = 0$

$x = 0 \quad x = 3 \quad x = -3$

$f(-3) = -(-3)^4 + 18(-3)^2 + 3$

$-81 + 18(9) + 3$

$-81 + 162 + 3$

$165 - 81$

x	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,3)$	3	$(3,\infty)$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$

$-4x(x-3)(x+3)$

$+ \quad - \quad -$

$+ \quad - \quad +$

increasing

$(-\infty,-3) \cup (0,3)$

decreasing

$(-3,0) \cup (3,\infty)$

relative max $(-3, 84)$

$(3, 84)$

relative min $(0, 3)$

4) $f(x) = \dfrac{x^2}{x-2}$

$f'(x) = \dfrac{2x(x-2) - x^2}{(x-2)^2} = 0$

$x \neq 2$

$2x^2 - 4x - x^2 = \dfrac{x^2 - 4x}{(x-2)^2}$

$\dfrac{x(x-4)}{(x-2)^2} = 0 \quad x = 0, 4$

x	$(-\infty,0)$	0	$(0,2)$	2	$(2,4)$	4	$(4,\infty)$
$f'(x)$	$+$	0	$-$	und	$-$	0	$+$

increasing $(-\infty,0) \cup (4,\infty)$

decreasing $(0,2) \cup (2,4)$

relative max $(0, 0)$

" min $(4, 8)$

5.4 Concavity and the Second Derivative Test

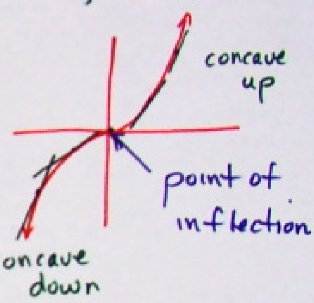
$f(x) = x^3$

$f'(x) = 3x^2 = 0$

$x = 0$

x	$(-\infty,0)$	0	$(0,\infty)$
$f'(x)$	$+$	0	$+$

increasing $(-\infty,0) \cup (0,\infty)$



$f''(x) = 6x = 0$

$x = 0$

x	$(-\infty,0)$	0	$(0,\infty)$
$f''(x)$	$-$	0	$+$

Test for Concavity

if $f''(x) > 0$, then $f(x)$ is concave up

if $f''(x) < 0$, then $f(x)$ is concave down.

Points of Inflection - any point where $f(x)$ changes concavity

$f''(x) = 0$ and $f''(x)$ must change sign.

$f(x) = x^4$

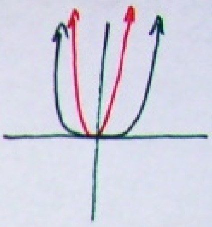
$f'(x) = 4x^3$

$f''(x) = 12x^2 = 0$

$x = 0$

x	$(-\infty,0)$	0	$(0,\infty)$
$f''(x)$	$+$	0	$+$

concave up $(-\infty,0)\cup(0,\infty)$



$f(x) = x^3 + 6x^2$ Determine the intervals of concavity and find all points of inflection

$f'(x) = 3x^2 + 12x$

$f''(x) = 6x + 12 = 0$

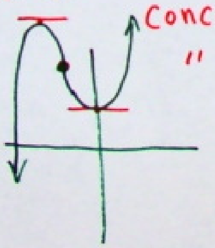
$x = -2$

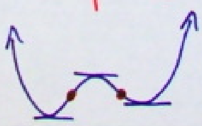
$f(-2) = -8 + 24$
16

x	$(-\infty,-2)$	-2	$(-2,\infty)$
$f''(x)$	$-$	0	$+$

concave down $(-\infty,-2)$
" up $(-2,\infty)$

inflection pt
$(-2, 16)$





$f(x) = \dfrac{24}{x^2+12} = 24(x^2+12)^{-1}$

$f'(x) = -24(x^2+12)^{-2}(2x)$
$= \dfrac{-48x}{(x^2+12)^2}$

$f''(x) = \dfrac{-48(x^2+12)^{\cancel{2}} - 2\cancel{(x^2+12)}(2x)(-48x)}{(x^2+12)^4} = 0$

$f''(x) = \dfrac{-48\cancel{(x^2+12)}\left[x^2+12-4x^2\right]}{(x^2+12)^3} = 0$

$48 \times 3 = 144$

$f''(x) = \dfrac{-48(-3x^2+12)}{(x^2+12)^3} = 0$

$= 144(x^2-4) = \dfrac{144(x-2)(x+2)}{(x^2+12)^3} = 0$

$x = 2 \quad x = -2$

x	$(-\infty,-2)$	-2	$(-2,2)$	2	$(2,\infty)$
$f''(x)$	$+$	0	$-$	0	$+$

concave up $(-\infty,-2)\cup(2,\infty)$
down $(-2,2)$

inflect pts $\left(-2, \frac{3}{2}\right)$ $\left(2, \frac{3}{2}\right)$

$\frac{24}{16}$

$$f(x) = x^{\frac{2}{3}} - x^{\frac{1}{3}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}}$$

$$f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} + \frac{2}{9}x^{-\frac{5}{3}} = 0$$

$$-x^{-\frac{4}{3}} + x^{-\frac{5}{3}} = 0$$

$$\frac{1}{x^{\frac{5}{3}}} = \frac{1}{x^{\frac{4}{3}}}$$

$$\frac{1}{\sqrt[3]{x^5}} = \frac{1}{\sqrt[3]{x^4}}$$

$$\frac{1}{x^5} = \frac{1}{x^4} \qquad x = 0$$

$$x^5 = x^4 \qquad x = 1$$

$$x^5 - x^4 = 0$$

$$x^4(x-1) = 0$$

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
$f''(x)$	$-$	0	$+$	0	$-$

$$f''(x) = -\frac{2}{9}x^{-4/3}\left(1 - x^{-1/3}\right)$$

$$= \underbrace{\frac{-2}{+9\sqrt[3]{x^4}}}_{\text{neg.}}\left(1 - \frac{1}{\sqrt[3]{x}}\right) \qquad \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$$

concave up $(0,1)$

" down $(-\infty,0) \cup (1,\infty)$

inflection pts $(0,0)$
$(1,0)$

Second Derivative Test

Suppose $f(x)$ is a continuous function in an interval containing $x = c$

- if $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a relative max point
- if $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a relative min point

* if $f''(c) = 0$ or $f''(c)$ does not exist, then the 2nd derivative test fails

ex11 locate all relative extrema for $f(x) = x^3 + 6x^2 - 15x - 90$

$$f'(x) = 3x^2 + 12x - 15 = 0$$
$$x^2 + 4x - 5 = 0$$
$$(x+5)(x-1) = 0$$
$$x = -5 \quad x = 1$$

$$f''(x) = 6x + 12 = 6(x+2)$$

$f''(-5) = $ neg

$x = -5$ rel max

$f''(1) = $ pos

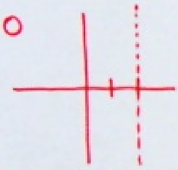
$x = 1$ rel min

5.5 Asymptotes and Intercepts

Vertical Asymptotes in a rational function, there will be a vertical asymptote (v.a.) at $x = c$ for any value of c in which the denominator is zero but the numerator is non-zero.

ex// $f(x) = \frac{x}{x-2}$ v.a $x=2$

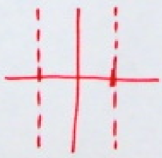
$x - 2 = 0$



$f(x) = \frac{-3}{x^2-1}$

$x^2-1=0$

$x=1 \quad x=-1$



$f(x) = \frac{4x}{x^2+4}$

$x^2+4=0$

no vert asymptote

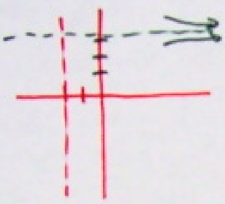
$f(x) = \frac{x-3}{x^2-9} = \frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)}$

$x^2-9=0 \quad x=3 \quad x=-3$

horizontal asymptotes $y = c$

to determine how a function behaves as x approaches a very large + or − value, it is necessary to use limits

$f(x) = \frac{3x^1}{1x+2}$



$\lim_{x\to\infty} \frac{3x^1}{x+2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{3}{1+\boxed{\cancel{\frac{2}{x}}}} = 3$

$y=3$

$\lim_{x\to\infty} \frac{2}{x} = 0 \qquad \frac{2}{100} \quad \frac{2}{1000}$

$\lim_{x\to-\infty} \frac{2}{x} = 3$

$f(x) = \frac{x}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{1}{x}}{1+\frac{1}{x^2}} \qquad \lim_{x\to\pm\infty} \frac{0}{1+0} = 0 \qquad y=0$

ex// $f(x) = \frac{5x^2-2x}{4x^2+1} \qquad y = \frac{5}{4}$

$f(x) = \frac{3x^0}{x^1-2} \qquad y=0$

<u>other asymptotes</u> if the degree of the numerator is higher than the degree of the denominator, a rational function has an <u>oblique</u> asymptote

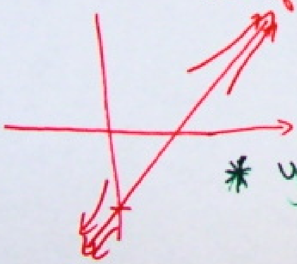
ex11 $f(x) = \dfrac{x^2-2x+3}{x+4}$

$$\begin{array}{r} x-6 \\ x+4\,\overline{)\,x^2-2x+3} \\ -\underline{(x^2+4x)} \\ -6x+3 \\ -\underline{(-6x-24)} \\ 27 \end{array}$$

$f(x) = x-6 \boxed{+\dfrac{27}{x+4}} \to 0$

$\lim\limits_{x\to\pm\infty} f(x)$ $\quad\lim\limits_{x\to\pm\infty}$

oblique (slant) asym $x-6$



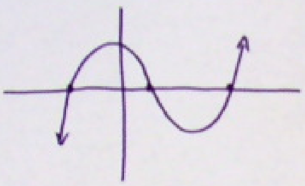
$\dfrac{x^3}{x-3}$ $\quad\dfrac{\sqrt{x^2+4}}{x-2}$

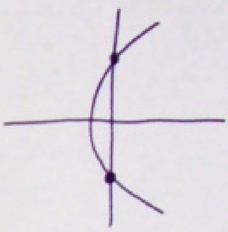
* you are not responsible for oblique asymptotes

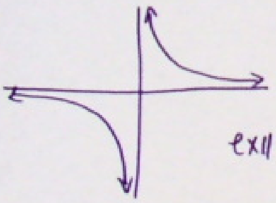
<u>Intercepts</u>

x-int
let $y=0$

y-int
let $x=0$







ex11 $y = \dfrac{3x+6}{x^2}$ $\qquad y = \dfrac{3(0)+6}{0^2}$

x-int

$\dfrac{0}{1} = \dfrac{3x+6}{x^2}$ $\quad 3x+6=0$

$x=-2$

$(-2,0)$

y-int

no y-int

5.6 Curve Sketching

1) analyze 1^{st} derivative
   - 1^{st} deriv test
   - intervals of increase / decrease
     - critical #'s
     - relative extrema

2) analyze 2^{nd} derivative
   - concavity test
   - intervals of concavity
   - inflection points

3) x and y intercepts

4) asymptotes - rational functions only

5) sketch graph

$f(x) = x^4 + 4x^3$

1) $f'(x) = 4x^3 + 12x^2 = 0$

$4x^2(x+3) = 0$

$x = 0 \quad x = -3$

x	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,\infty)$
$f'(x)$	$-$	0	$+$	0	$+$

increasing $(-3,0) \cup (0,\infty)$

decreasing $(-\infty,-3)$

relative min $(-3,-27)$

" max none

2) $f''(x) = 12x^2 + 24x = 0$

$12x(x+2) = 0$

$x = 0 \quad x = -2$

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,\infty)$
$f''(x)$	$+$	0	$-$	0	$+$

concave up $(-\infty,-2) \cup (0,\infty)$

" down $(-2,0)$

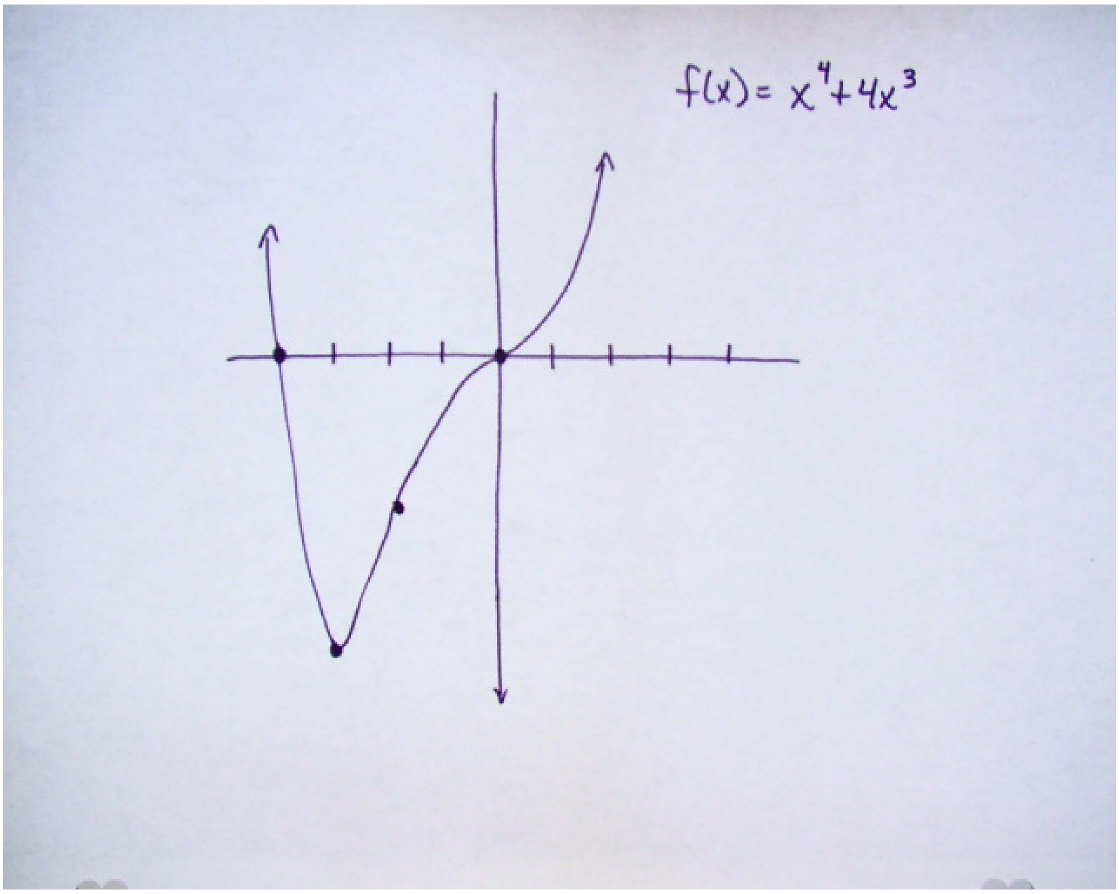
inflection pts $(-2,-16)$, $(0,0)$

3) x-int $0 = x^3(x+4)$

$x = 0 \quad x = -4$

y-int $y = 0$

4) no asymptotes



$$f(x) = \frac{2x}{x-3}$$

1) $f'(x) = \dfrac{2(x-3)-2x}{(x-3)^2}$

$= \dfrac{-6}{(x-3)^2} = \dfrac{0}{1}$

$-6=0$ no sol

no critical points

x	$(-\infty,3)$	3	$(3,\infty)$
$f'(x)$	$-$	und	$-$

decreasing $(-\infty,3)\cup(3,\infty)$

increasing never

no relative extrema

2) $f'(x) = -6(x-3)^{-2}$

$f''(x) = 12(x-3)^{-3}(1)$

$= \dfrac{12}{(x-3)^3} = 0$

no critical points

x	$(-\infty,3)$	3	$(3,\infty)$
$f''(x)$	$-$	und	$+$

concave up $(3,\infty)$

down $(-\infty,3)$

no inflection pts

3) x-int $\quad$ y-int

$\dfrac{0}{1} = \dfrac{2x}{x-3}$ $\quad$ $y=0$

$x=0$

4) v.a.

$x=3$

$\lim_{x\to3^+} f(x) = \infty$

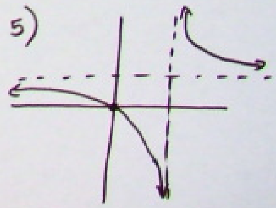
$\lim_{x\to3^-} f(x) = -\infty$

h.a. $y=2$

$\lim_{x\to\infty} f(x) = 2^+$

$\lim_{x\to-\infty} f(x) = 2^-$

5)



$f(x) = x^{\frac{2}{3}} + \frac{1}{5}x^{5/3}$

1) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}} = 0$

$\frac{1}{3}x^{-\frac{1}{3}}\left[2 + x^{1}\right] = 0$

$\frac{2+x}{3\sqrt[3]{x}} = 0$

$x = -2$ $x = 0$

horiz vertical

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,\infty)$
$f'(x)$	+	0	−	und	+
	/		\		/

increasing $(-\infty,-2) \cup (0,\infty)$
decreasing $(-2,0)$
rel max $(-2, 0.95)$
min $(0,0)$

2) $f''(x) = -\frac{2}{9}x^{-4/3} + \frac{2}{9}x^{-1/3}$

$= -\frac{2}{9}x^{-4/3}\left[1 - x\right]$

$= \frac{-2(1-x)}{9\sqrt[3]{x^4}} = 0$

$x = 1$ $x = 0$

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
$f''(x)$	−	und	−	0	+

Concave down $(-\infty,0) \cup (0,1)$
up $(1,\infty)$
inflection pt $(1, \frac{6}{5})$

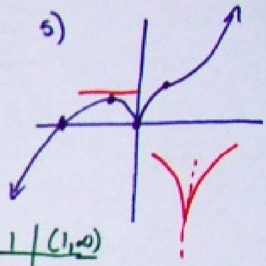
3) $0 = x^{\frac{2}{3}} + \frac{1}{5}x^{\frac{5}{3}}$

$x^{\frac{2}{3}}(1 + \frac{1}{5}x)$

$x = 0$ $x = -5$ $y = 0$

4) no asymptotes

5)



5.7 Chapter Review

1) find f' and f'' of a function
- polynomial
- rational

2) find critical #'s of a function
$f'(x) = 0$ or $f'(x)$ does not exist

3) find absolute extrema $[a,b]$

4) 1^{st} derivative test
- increasing / decreasing intervals
- relative extrema

5) concavity test $f''(x)$
- intervals of concavity
- inflection points

6) 2^{nd} derivative test for local extrema

7) find equation of asymptotes

* 8) analyze a function
1) 1^{st} deriv
2) 2^{nd} deriv
3) intercepts
4) asymptotes
5) sketch