4.3 Slope of a Tangent Line at a Specific Point

4.1 review math 10 * slope formula

4.2 not important - omit

* goal of differential Calculus is to find the slope of a tangent line to a function

$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta Y}{\Delta X} = \frac{y_2 - y_1}{x_2 - x_1}$

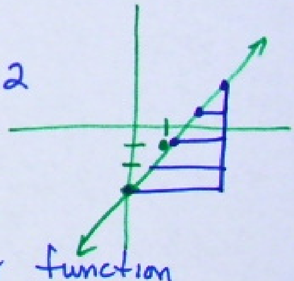
$m = \frac{-3-1}{0-2} = \frac{-4}{-2} = 2$

ex1 $f(x) = \overset{m}{\boxed{2}}x - 3$

$x_1 = 2 \quad y_1 = 1 \quad (2,1)$

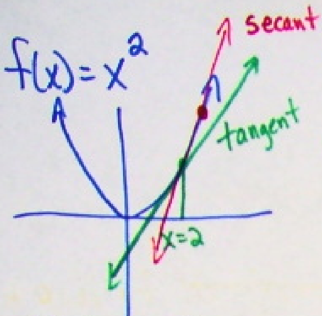
$x_2 = 0 \quad y_2 = -3$

$(0,-3)$



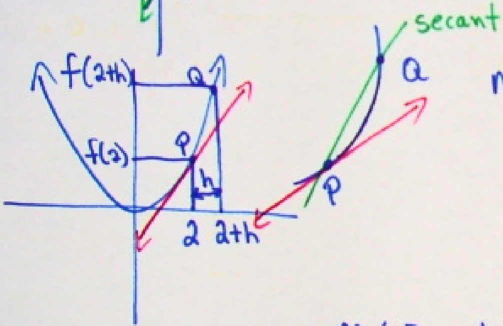
* slope of a linear function is independent of the 2 points selected.

$f(x) = x^2$



find slope of f(x) at x=2 (tangent line)

$m = \frac{y_2 - y_1}{x_2 - x_1}$



$$m_{\overleftrightarrow{PQ}} = \frac{\overset{y_2}{f(2+h)} - \overset{y_1}{f(2)}}{\cancel{2}+h-\cancel{2}}$$

$$= \frac{(2+h)^2 - (2)^2}{h}$$

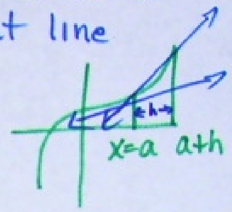
$$= \frac{\cancel{4}+4h+h^2-\cancel{4}}{h} = \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$m_{\overleftrightarrow{PQ}} = 4+h$

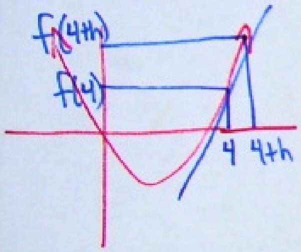
$\lim_{h \to 0} 4+h = 4 \leftarrow$ slope of the tangent line

* the slope of the tangent line to $f(x)$ at $x=a$ is the limiting slope of the secant line between $x=a$ and $x=a+h$

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$



ex) find slope of tangent line to $f(x) = x^2 - 3x$ at $x=4$



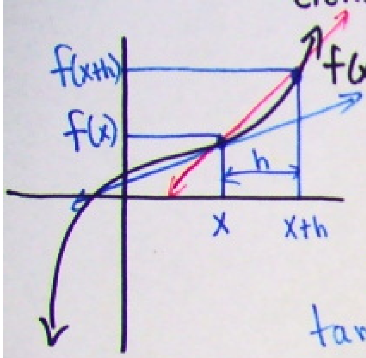
$$m = \lim_{h \to 0} \frac{\overbrace{\left[(4+h)^2 - 3(4+h)\right]}^{f(4+h)} - \overbrace{[4]}^{f(4)}}{h}$$

limit of indeterminate form

$$\lim_{h \to 0} \frac{\cancel{16} + 8h + h^2 - \cancel{12} - 3h - \cancel{4}}{h}$$

$$\frac{5h + h^2}{h} = \frac{\cancel{h}(5 + \cancel{h})}{\cancel{h}} = 5$$

4.4 Slope of a Tangent Line at a General Point $(x, f(x))$



$$\boxed{m = \lim_{h \to 0} \frac{f(\overset{y_2}{x+h}) - f(\overset{y_1}{x})}{h}}$$

delta

$$\frac{\Delta y}{\Delta x}$$

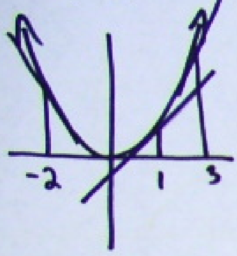
Derivative of $f(x)$ is the slope of the tangent line

$f'(x)$ or $\frac{dy}{dx}$

f prime

* summary pg 174 D_x

$f(x)=x^2$, find the derivative



$m=\frac{y_2-y_1}{x_2-x_1}$

$$* \boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+\cancel{h})}{\cancel{h}} = \boxed{2x}$$

$f'(1)=2$

$f'(3)=6$ $\quad f'(-2)=-4$

$y=\sqrt{x+2}$ find $\frac{dy}{dx}=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\left[\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}\right]$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{x+\cancel{h}^{1}+2-(x+2)}{\cancel{h}_{1}\left[\sqrt{x+h+2}+\sqrt{x+2}\right]} = \frac{1}{\sqrt{x+\cancel{h}+2}+\sqrt{x+2}} =$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x+2}} \checkmark$$

4.5 Elementary Differentiation Rules

* the process of <u>differentiation</u> is determining the derivative of a function

recall: the derivative is defined as:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

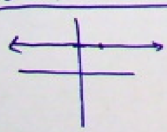
$f(x)=x^2$ $\quad f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}=\frac{\cancel{h}(2x+h)}{\cancel{h}}$

$=2x$

$f(x)=x^{20}$

$f(x)=(3x^2+5x)^7(2x-x^3)^5$

<u>Constant Rule</u> $\quad f(x)=c$ where c is a constant



then $f'(x)=0$

Power Rule

if $f(x) = x^n$, then $f'(x) = nx^{n-1}$

ex// $f(x) = x^2$
$f'(x) = 2x$

$y = x^{10}$
$\frac{dy}{dx} = 10x^9$

$y = \sqrt[5]{x}$
$y = x^{\frac{1}{5}}$
$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$
$= \frac{1}{5\sqrt[5]{x^4}}$

Constant Multiple Rule

if $f(x) = c \cdot g(x)$ then $f'(x) = c \cdot g'(x)$

$f(x) = 6x^3 \quad f'(x) = 6(3x^2) = 18x^2$

Extended Power Rule

if $f(x) = cx^n$
then $f'(x) = ncx^{n-1}$

ex// $y = 4x^6 \quad \frac{dy}{dx} = 24x^5$

Sum and Difference Rule

if $f(x) = p(x) + q(x) + r(x) + \ldots$
then $f'(x) = p'(x) + q'(x) + r'(x) + \ldots$

$y = x^5 + 3x^2 - 5x^1 + 2$
$\frac{dy}{dx} = 5x^4 + 6x - 5$

$y = \frac{3}{x} - 5\sqrt{x}$
$y = 3x^{-1} - 5x^{\frac{1}{2}}$
$\frac{dy}{dx} = -3x^{-2} - \frac{5}{2}x^{-\frac{1}{2}}$

$f(x) = (x+4)(2x-5)$
$f(x) = 2x^2 + 3x - 20$
$f'(x) = 4x + 3$

4.6 Product Rule

$f(x) = x^2 + x^3$

$f'(x) = 2x + 3x^2$

$f(x) = (x^2)(x^3) = f(x) = x^5$

$f'(x) = (2x)(3x^2)$ $\quad f'(x) = 5x^4$

$= 6x^3$ wrong!

<u>Product Rule</u> if $f(x) = p(x) \cdot q(x)$

then $f'(x) = p'(x) \cdot q(x) + q'(x) \cdot p(x)$

$f(x) = \underset{p}{x^2} \cdot \underset{q}{x^3}$ $\quad f'(x) = 2x \cdot x^3 + 3x^2 \cdot x^2$

$2x^4 + 3x^4 = 5x^4$

$f(x) = \underbrace{(2x+1)}_{p} \cdot \underbrace{(x^2-3x)}_{q}$ $\quad f'(x) = 2(x^2-3x) + (2x-3)(2x+1)$

extended product rule

if $f(x) = p(x) \cdot q(x) \cdot r(x)$

then $f'(x) = p'(x)\, q(x)\, r(x) + q'(x)\, p(x)(r(x)) + r'(x)\, p(x)\, q(x)$

ex!! find the slope of the tangent line to

$y = \underbrace{(x^2+3x)}_{p}\underbrace{(x^3-6x+1)}_{q}$ at $x = 2$

$\left.\frac{dy}{dx}\right| = (2x+3)(x^3-6x+1) + (3x^2-6)(x^2+3x)$

$\left.\frac{dy}{dx}\right|_{x=2}$ $(7)(-3) + (6)(10)$

$= 39$

4.9 Combining Differentiation Rules

* practice! practice! practice!

1) $y = \dfrac{2x^3}{(4x-1)^4}$ find the derivative

$$\frac{dy}{dx} = \frac{6x^2(4x-1)^4 - 4(4x-1)^3(4)(2x^3)}{(4x-1)^8}$$

$$= \frac{2x^2\cancel{(4x-1)^3}\left[3(4x-1) - 16x\right]}{(4x-1)^{\cancel{8}\,5}}$$

$$= \frac{2x^2(-4x-3)}{(4x-1)^5} = -\frac{2x^2(4x+3)}{(4x-1)^5}$$

4.9 Combining Differentiation Rules

2) $f(x) = \overset{p}{(2x+3)}\overset{q}{(\sqrt{5x-1})}$ if $p \cdot q$ then

$f'(x) = 2(5x-1)^{1/2} + \frac{1}{2}(5x-1)^{-1/2}(5)(2x+3)$ $p'q + q'p$

3) $y = \dfrac{3}{(2x+1)^5} = 3(2x+1)^{-5}$

$$\frac{dy}{dx} = -15(2x+1)^{-6}(2) = \frac{-30}{(2x+1)^6}$$

4) $f(x) = \sqrt{x}\,(x^2 + x^3)$

$= x^{\frac{1}{2}}(x^2 + x^3) = x^{\frac{5}{2}} + x^{\frac{7}{2}}$

$f'(x) = \frac{5}{2}x^{\frac{3}{2}} + \frac{7}{2}x^{\frac{5}{2}} = \frac{5}{2}\sqrt{x^3} + \frac{7}{2}\sqrt{x^5}$

4.10 Implicit Differentiation

implicit
- not obvious

$x^2+y^2=25$

$xy^3+y^2+3x=6$

explicit

$y=x^2$

$y=3x+1$

* Key - chain rule

$y=x^2$

$\frac{dy}{dx}=2x$

$y=x^2 \quad y \Leftarrow x \Leftarrow t$

$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$

$=2x\boxed{\frac{dx}{dt}}$

compensating term

$y=x^2$

$\frac{dy}{dp}=2x\frac{dx}{dp}$

4.10 Implicit Differentiation

$\frac{d(x^2)}{dx}=2x \qquad \frac{d(x^2)}{dt}=2x\frac{dx}{dt} \qquad \frac{d(y^2)}{dx}=2y\frac{dy}{dx}$

If $x^2+y^2=25$, find $\frac{dy}{dx}$

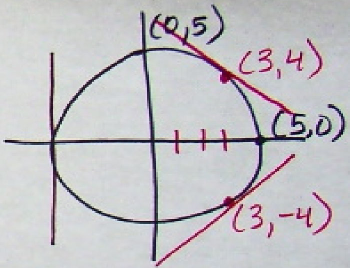
$\frac{d(x^2)}{dx}+\frac{d(y^2)}{dx}=\frac{d(25)}{dx}$

$2x+2y\frac{dy}{dx}=0$

* isolate $\frac{dy}{dx}$

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}$ $\Leftarrow$ $y\neq 0$



4.10 Implicit Differentiation

$x^2 + 3xy - y^3 = 9$ find $\frac{dy}{dx}$

$$\frac{d(x^2)}{dx} + \frac{d(3xy)}{dx} - \frac{d(y^3)}{dx} = \frac{d(9)}{dx}$$

$$2x + \underbrace{3y + \frac{dy}{dx}(3x)} - 3y^2\frac{dy}{dx} = 0$$

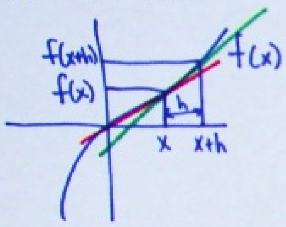
$$2x + 3y = 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx}$$

$$2x + 3y = \frac{dy}{dx}\left(3y^2 - 3x\right)$$

$$\frac{dy}{dx} = \frac{2x+3y}{3y^2 - 3x}$$

Chapter 4.11 Review

1) definition of a derivative



$m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

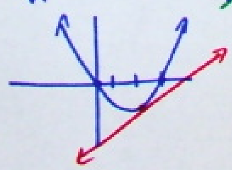
ex1 $f(x) = x^2 - 3x$ find $f'(x)$ find $f'(2)$

$$f'(x) = \lim_{h \to 0} \frac{\left[(x+h)^2 - 3(x+h)\right] - \left[x^2 - 3x\right]}{h} = \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \frac{\cancel{h}(2x + h - 3)}{\cancel{h}}$$

$$\boxed{f'(x) = 2x - 3}$$

$f'(2) = 1$



2) Differentiate Functions using Rules

a) do not simplify

$y = 3x^4 + 5x^2 - 2$

$\frac{dy}{dx} = 12x^3 + 10x$

$f(x) = (3x-2)^6$

$f'(x) = 6(3x-2)^5(3)$

$y = \frac{x^2 - 2x}{3x+1}$

$\frac{dy}{dx} = \frac{(2x-2)(3x+1) - 3(x^2-2x)}{(3x+1)^2}$

$y = 5\sqrt{2x-1} = 5(2x-1)^{1/2}$

$\frac{dy}{dx} = 5\left(\frac{1}{2}\right)(2x-1)^{-1/2}(2)$

b) simplest form - factored (X)

$y = (\overset{p}{2x-1})^3(\overset{q}{5x+3})^4$ $p \cdot q$ $p'q + q'p$

$\frac{dy}{dx} = 3(2x-1)^2(2)(5x+3)^4 + 4(5x+3)^3(5)(2x-1)^3$

$= 2(2x-1)^2(5x+3)^3\left[3(5x+3) + 10(2x-1)\right]$

$= (2)(2x-1)^2(5x+3)^3(35x-1)$ simplest factored form

3) applications

$\overset{m=0}{\longleftrightarrow}$

find the points where the tangent line is horizontal to

$y = \frac{x^2-1}{x^2+1}$ $\frac{dy}{dx} = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = 0$ $\frac{a}{b} = 0$ $a = 0$

$\frac{dy}{dx} = 0$ $\frac{2x[x^2+1-x^2+1]}{\cancel{(x^2+1)^2}} = 0$ $\frac{2x(2)}{(\ ?\ )} = 0$ $x = 0$

$(0, -1)$

4) implicit differentiation

find $\frac{dy}{dx}$ for $x^3 + x^2y - 3y^2 = 2$ implicit

$$\frac{d(x^3)}{dx} + \frac{d(x^2y)}{dx} - \frac{d(3y^2)}{dx} = \frac{d(2)}{dx}$$

$$\underbrace{3x^2 + 2xy} + (1)\frac{dy}{dx}x^2 - 6y\frac{dy}{dx} = 0$$

$$3x^2 + 2xy = 6y\frac{dy}{dx} - x^2\frac{dy}{dx}$$

if $a=b$ then $b=a$

$$= \frac{dy}{dx}\left[6y - x^2\right]$$

$$\frac{dy}{dx} = \frac{3x^2 + 2xy}{6y - x^2}$$