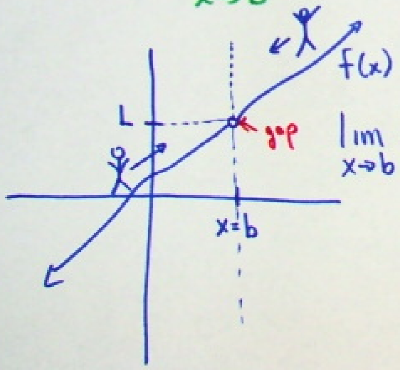
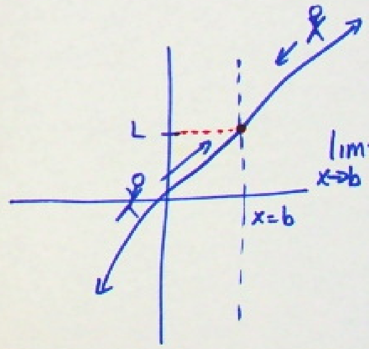
3.2 Limits - A Visual Approach

Definition: if as x <u>approaches</u> b from both the right and left, f(x) approaches the single real number L, then L is called the limit of the function f(x) as x approaches b, and we write $\lim_{x \to b} f(x) = L$



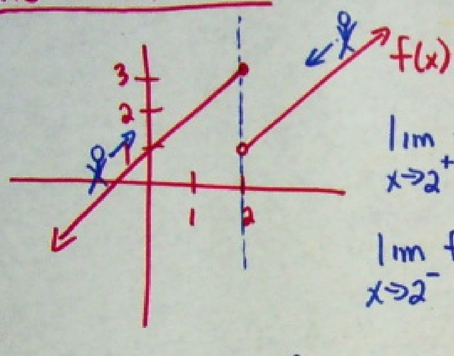
$\lim_{x \to b} f(x) = L$



$\lim_{x \to b} f(x) = L$

3.2 Limits - A Visual Approach

<u>one-sided limits</u>



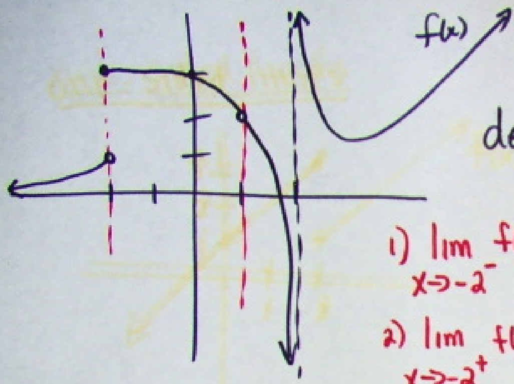
$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x)$ does not exist

$\lim_{x \to 2^-} f(x) = 3$

* in order for the $\lim_{x \to b} f(x)$ to exist, the one-sided limits must be the same

ie: $\lim_{x \to b^+} f(x) = \lim_{x \to b^-} f(x) = L$ then $\lim_{x \to b} f(x)$ exists $= L$



determine the following limits

1) $\lim_{x \to -2^-} f(x) = 1$

2) $\lim_{x \to -2^+} f(x) = 3$

3) $\lim_{x \to -2} f(x)$ does not exist

4) $\lim_{x \to 1^+} f(x) = 2$

5) $\lim_{x \to 1^-} f(x) = 2$

6) $\lim_{x \to 1} f(x) = 2$

7) $\lim_{x \to 2^-} f(x) = -\infty$

8) $\lim_{x \to 2^+} f(x) = \infty$

9) $\lim_{x \to 2} f(x)$ does not exist

10) $\lim_{x \to \infty} f(x) = \infty$

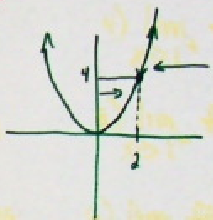
11) $\lim_{x \to -\infty} f(x) = 0$

3.3 Strategies for Evaluating Limits

* the $\lim_{x \to b} f(x)$ is only concerned with the behaviour of $f(x)$ <u>near</u> $x = b$ not at $x = 6$

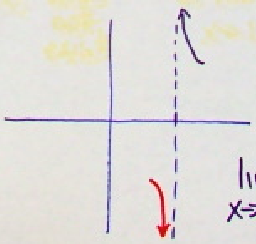
1) direct substitution

$$\lim_{x \to 2} x^2 = (2)^2 = 4$$



2) sign analysis

$$\lim_{x \to 3} \frac{x+2}{x-3} = \frac{5}{0}$$



$\lim_{x \to 3^+} f(x) = \frac{+}{+} = \infty$

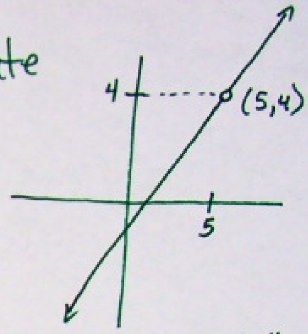
$\lim_{x \to 3^-} f(x) = \frac{+}{-} = -\infty$

$\lim_{x \to 3} f(x)$ does not exist

3) factoring

$$\lim_{x\to5}\frac{x^2-6x+5}{x-5}=\frac{0}{0}$$ indeterminate form

$$\frac{\cancel{(x-5)}(x-1)}{\cancel{(x-5)}}=4$$



4) simplify

$$\lim_{x\to0}\frac{(3+x)^2-4(3+x)+3}{x}=\frac{0}{0}$$

$$\frac{\cancel{9}+6x+x^2-\cancel{12}-4x+\cancel{3}}{x}$$

$$\frac{2x+x^2}{x}=\frac{\cancel{x}(2+x)}{\cancel{x}}=2$$

$$\lim_{m\to10}\frac{\frac{3}{3}\left(\frac{2}{m-4}\right)-\left(\frac{1}{3}\right)\frac{m-4}{m-4}}{m-10}$$

$$\frac{6-(m-4)}{3(m-4)}\cdot\frac{1}{(m-10)}$$

$$\frac{-1\,\cancel{10-m}}{3(m-4)\cancel{(m-10)}}=\frac{-1}{18}$$

5) rationalization

$$\lim_{r\to6}\frac{\sqrt{3+r}-3}{r-6}\cdot\frac{[\sqrt{3+r}+3]}{[\sqrt{3+r}+3]}=\frac{0}{0}$$

$$\sqrt{2}\times\sqrt{2}=\sqrt{2^2}=2$$

$$(2+\sqrt{3})(2-\sqrt{3})=4-3=1$$

$$\frac{3+r-9}{(r-6)[\sqrt{3+r}+3]}=\frac{\cancel{(r-6)}\quad 1}{\cancel{(r-6)}[\sqrt{3+r}+3]}=\frac{1}{6}$$

6. limits at infinity of rational functions

$$f(x)=\frac{p(x)}{q(x)}\qquad *\ \lim_{x\to\infty}\frac{b}{x^n}=0\qquad \frac{1}{x^2}\quad \frac{1}{100}\quad \frac{1}{10000}$$

$$\lim_{x\to\infty}\frac{6x-5}{2x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\frac{6-\cancel{\boxed{\frac{5}{x}}}}{2+\cancel{\boxed{\frac{3}{x}}}}=3$$

$$\lim_{x\to\infty}\frac{3x^2}{x^3-4x}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$\frac{\boxed{\frac{3}{x}}\,0}{1-\boxed{\frac{4}{x^2}}\,0}=\frac{0}{1}=0$$

7) ~~not~~ responsible for - optional

8) absolute value $\quad * \; |b| = \begin{cases} b & \text{if } b \geq 0 \\ -b & \text{if } b < 0 \end{cases}$

$$\lim_{x\to 2} \frac{|x^2-x-2|}{x-2} = \frac{|(x-2)(x+1)|}{x-2} = \begin{cases} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}, & x>2 \\ -\dfrac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}, & x<2 \end{cases} = \begin{cases} x+1, & x>2 \\ -(x+1), & x<2 \end{cases}$$

$\lim_{x\to 2^+} f(x) = 3 \qquad \lim_{x\to 2^-} f(x) = -3 \qquad \lim_{x\to 2} f(x) =$ does not exist

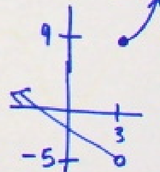
9) piecewise functions

$$\lim_{x\to 3} \begin{cases} x^2, & x \geq 3 \\ -2x+1, & x<3 \end{cases}$$

$\lim_{x\to 3^+} f(x) = 9$

$\lim_{x\to 3^-} f(x) = -5$

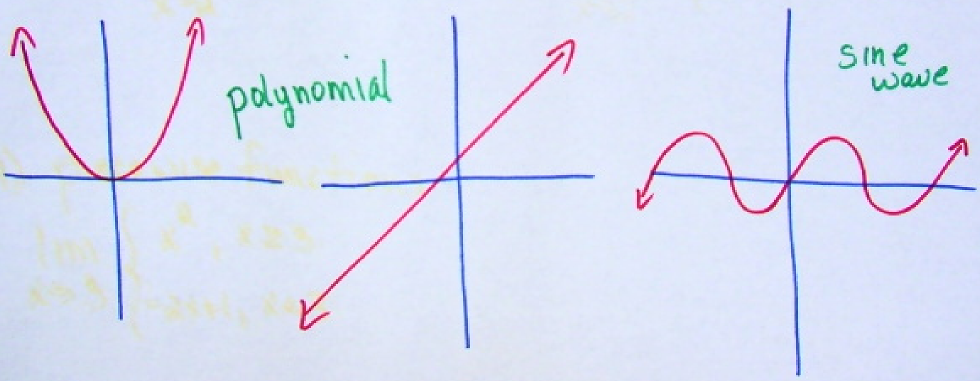
$\lim_{x\to 3} f(x)$ does not exist



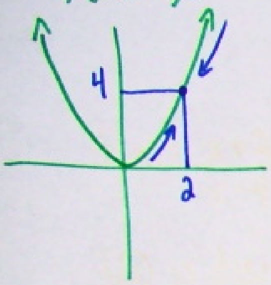
3.4 Continuity of a Function

Informally, the concept of continuity is a simple one. A function is continuous if you can draw it without lifting your pencil.

ie: no gaps, asymptotes etc!



$f(x) = x^2$



$\lim_{x\to 2} f(x) = 4$

$\lim_{x\to 2^+} f(x) = 4$

$\lim_{x\to 2^-} f(x) = 4$

* note: $f(2) = 4$

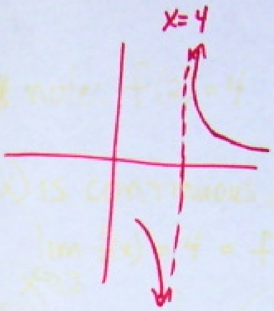
$f(x)$ is continuous at $x=2$
since $\lim_{x\to 2} f(x) = 4 = f(2)$

definition: a function is continuous at $x = b$
if $\lim_{x\to b} f(x) = f(b)$

Classifying Discontinuities

1) infinite discontinuity

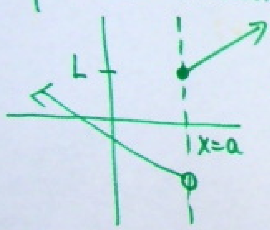
$f(x) = \dfrac{x+2}{x-4}$ let $x=4$ $\dfrac{6}{0}$



$f(x)$ is discontinuous at $x=4$
infinite discontinuity $\dfrac{b}{0}$

$\lim_{x\to 4} f(x) \neq f(4)$

2) jump discontinuity.



$f(x) = \begin{cases} \_\_\_\_ \\ \_\_\_\_ \end{cases}$

$f(a) = L$

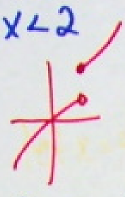
$\lim_{x\to a} f(x)$ does not exist

$$f(x)=\begin{cases} x^2, & x\ge 2 \\ x+1, & x<2\end{cases}$$

$\lim_{x\to 2^+} f(x) = 4$

$\lim_{x\to 2^-} f(x) = 3$

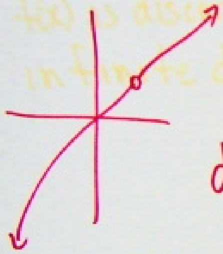
$f(2)=4$



$\lim_{x\to 2} f(x)$ does not exist

jump discont at $x=2$

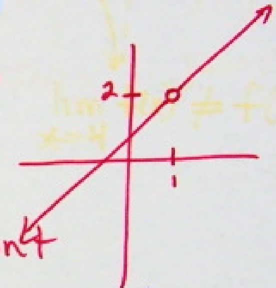
3) removable discontinuity $\left(\frac{0}{0}\right)$

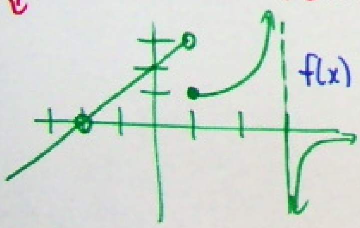


$f(x)=\frac{x^2-1}{x-1}=\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$

discont at $x=1$

removable discont





$x=-2$ removable

$x=1$ jump

$x=3$ infinite

determine and classify the discontinuity

1) $f(x)=\frac{x^2-4}{x-2}$ $x=2$ $\frac{0}{0}$ $\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$ removable discontinuity

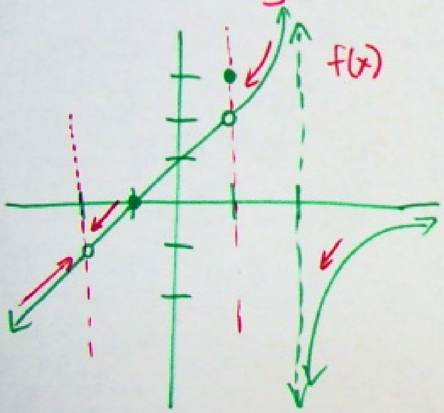
2) $f(x)=\begin{cases} 3x+1, & x\ge 1 \\ -2x+3, & x<1\end{cases}$ $\lim_{x\to 1^+} f(x)=4$ $\lim_{x\to 1^-} f(x)=1$ $x=1$ jump

3) $f(x)=\frac{x}{x-2}$ $x=2$ $\frac{2}{0}$ infinite

3.5 Chapter Review

Chapter 3 Exam

1) determine the value of a function and limits associated with the function from a diagram (3.2)



a) $f(-1) = 0$

b) $f(1) = 3$

c) $f(2)$ is undefined

d) $\lim_{x \to -2^-} f(x) = -1$

e) $\lim_{x \to -2^+} f(x) = -1$

f) $\lim_{x \to -2} f(x) = -1$

g) $\lim_{x \to 1^+} f(x) = 2$

h) $\lim_{x \to 1^-} f(x) = 2$

i) $\lim_{x \to 1} f(x) = 2$

j) $\lim_{x \to 2^+} f(x) = -\infty$

k) $\lim_{x \to \infty} f(x) = 0$

l) $\lim_{x \to -\infty} f(x) = -\infty$

2) determine limits using different strategies (3.3)

a) $\lim_{x \to 0} \frac{6x}{x^2+3x} = \frac{\cancel{x}(6)}{\cancel{x}(x+3)} = 2$

b) $\lim_{x \to 3^-} \frac{2x}{x-3} \quad \frac{+}{-} = -\infty$

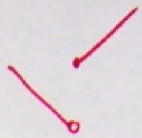
c) $\lim_{x \to \infty} \frac{x+1}{x-2} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{1+\frac{1}{x}}{1-\frac{2}{x}} = 1$

d) $\lim_{x \to 1} \frac{2-\sqrt{5-x}}{1-x} \left[\frac{2+\sqrt{5-x}}{2+\sqrt{5-x}}\right] = \frac{4-(5-x)}{} = \frac{-1+x}{(1-x)(2+\sqrt{5-x})}$

$\frac{-\cancel{(1-x)}}{\cancel{(1-x)}(2+\sqrt{5-x})} = \frac{-1}{4}$

3) piecewise functions (3.3)

$$f(x) = \begin{cases} x+1 & , x \geq 3 \\ -x+2 & , x < 3 \end{cases}$$



a) $\lim_{x \to 3^+} f(x)$

4

b) $\lim_{x \to 3^-} f(x)$

-1

c) $\lim_{x \to 3} f(x)$

does not exist

4) locate and classify discontinuities (3.4)

a) $f(x) = \frac{x^2 - x - 6}{x-3} = \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}}$ removable $x=3$

b) $f(x) = \frac{x+5}{x-3}$ infinite $x=3$